

MIXED FINITE ELEMENT FORMULATION FOR NON-ISOTHERMAL POROUS MEDIA IN DYNAMICS

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Abstract. We present a mixed finite element formulation for the spatial discretization in dynamic analysis of non-isothermal variably saturated porous media using different order of approximating functions for solid displacements and fluid pressures/temperature. It is known in fact that there are limitations on the approximating functions N_u and N_p for displacements and pressures if the Babuska-Brezzi convergence conditions or their equivalent [1] are to be satisfied. Although this formulation complicates the numerical implementation compared to equal order of interpolation, it provides competitive advantages e.g. in speed of computation, accuracy and convergence.

1 INTRODUCTION

The numerical modelling of the dynamic response of non-isothermal multiphase geomaterials is of great interest in many fields, including Environmental Geomechanics, Structural Mechanics and Biomechanics. Based on the improved efficiency and stability of appropriate numerical methods, and the increased performance of the computational equipments, the efficient numerical treatment of coupled multi-field problems has become possible in the last decades. Effective numerical simulations require a mixed element (e.g. Taylor-Hood element [2]) with different order of approximating functions for solid displacements and fluid pressures/temperature. Consequently, numerical complexities arise in the solutions because the basis functions in the u - p (solid-displacements and pressure) formulation are of different order (e.g. second order for displacements, first order for pressure, to obtain the same kind of approximation for effective stress and fluid pressures). In the use of mixed elements it has been reported by several authors [3, 4, 5, 6] that inaccuracies in pressure estimates can result from spurious spatial fluctuations of estimated

pore pressure between adjacent elements. This is especially likely to occur at permeable boundaries in the initial time-marching steps, or when short time steps are used to represent rapidly changing of fluid phases or transient applied loads. Moreover, Murad et al. [7] noted that, even with Taylor-Hood elements, potentially unstable results may occur when there are discontinuous initial conditions at the initiation of a time-marching solution.

This work presents, as a novel contribution, a mixed finite element formulation for fully coupled non-isothermal multiphase geomaterials in dynamics.

The multiphase model is developed following Lewis and Schrefler [8]. The u - p - T formulation is obtained by neglecting the relative fluids acceleration and their convective terms, which is valid for low frequency problems as in earthquake engineering [9]. In the model devolvement, the dynamic seepage forcing terms in the mass balance equations and in the enthalpy balance equation and the compressibility of the grain at the microscopic level are neglected. The Taylor-Hood approach is applied to discretize the governing equations in space, while the generalized Newmark scheme is used for the time discretization. The final non-linear set of equations is solved by the Newton-Raphson method with a monolithic approach [8, 10]. The model has been implemented in the finite element code COMES-GEO, [8, 11, 12, 13, 14, 15, 16, 17, 18].

The implemented model is validated through the comparison with analytical or finite element quasi-static or dynamic solutions.

2 MACROSCOPIC BALANCE EQUATIONS

The full mathematical model necessary to simulate the thermo-hydro-mechanical behaviour of partially saturated porous media was developed within the Hybrid Mixture Theory by Lewis and Schrefler [8], using averaging theories according to Hassanizadeh and Gray [19, 20]. After neglecting the acceleration of the fluids in the governing equations of Lewis and Schrefler [8], a set of balance equations for the whole multiphase medium is obtained and presented in this paper.

Linear momentum balance equations of the mixture

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{g} - \rho \mathbf{a}^s = 0 \quad (1)$$

Dry air mass balance equation

$$\begin{aligned} & \operatorname{div} \left(\rho^{ga} \frac{k^{rg} \mathbf{k}^g}{\mu^g} [-\operatorname{grad} p^g + \rho^g \mathbf{g}] \right) + \operatorname{div} \left(\rho^g \frac{M_a M_w}{M_g^2} \mathbf{D}_g^{ga} \operatorname{grad} \left(\frac{p^{gw}}{p^g} \right) \right) \\ & + \rho^{ga} S_g \operatorname{div} \mathbf{v}^s + n S_g \dot{\rho}^{ga} - \rho^{ga} n \dot{S}_w - \rho^{ga} \beta_s [1 - n] S_g \dot{T} = 0 \end{aligned} \quad (2)$$

Water species (liquid and vapour) mass balance equation

$$\begin{aligned} & \operatorname{div} \left(\rho^w \frac{k^{rw} \mathbf{k}^w}{\mu^w} [-\operatorname{grad} (p^g - p^c) + \rho^w \mathbf{g}] \right) + \operatorname{div} \left(\rho^{gw} \frac{k^{rg} \mathbf{k}^g}{\mu^g} [-\operatorname{grad} p^{gw} + \rho^{gw} \mathbf{g}] \right) \\ & - \operatorname{div} \left(\rho^g \frac{M_a M_w}{M_g^2} \mathbf{D}_g^{gw} \operatorname{grad} \left(\frac{p^{gw}}{p^g} \right) \right) + [\rho^w S_w + \rho^{gw} S_g] \operatorname{div} \mathbf{v}^s + n [\rho^w - \rho^{gw}] \dot{S}_w \\ & - [\rho^w \beta_{sw} + \rho^{gw} \beta_s [1 - n] S_g] \dot{T} + n S_g \dot{\rho}^{gw} + \rho^w \frac{n S_w}{K_w} [\dot{p}^g - \dot{p}^c] = 0 \end{aligned} \quad (3)$$

Enthalpy balance equation for the multiphase medium

$$\begin{aligned}
 & -\operatorname{div} \left(\rho^w \frac{k^{rw} \mathbf{k}}{\mu^w} [-\operatorname{grad} (p^g - p^c) + \rho^w \mathbf{g}] \right) \Delta H_{vap} - \operatorname{div} (\chi_{eff} \operatorname{grad} T) - \rho^w S_w \operatorname{div} \mathbf{v}^s \Delta H_{vap} \\
 & + \left[C_p^w \rho^w \frac{k^{rw} \mathbf{k}}{\mu^w} [-\operatorname{grad} (p^g - p^c) + \rho^w \mathbf{g}] + C_p^g \rho^g \frac{k^{rg} \mathbf{k}}{\mu^g} [-\operatorname{grad} p^g + \rho^g \mathbf{g}] \right] \cdot \operatorname{grad} T \\
 & + (\rho C_p)_{eff} \dot{T} - \rho^w \frac{n S_w}{K_w} [\dot{p}^g - \dot{p}^c] \Delta H_{vap} + \beta_{sw} \dot{T} \Delta H_{vap} - n [\rho^w - \rho^{gw}] \dot{S}_w \Delta H_{vap} = 0
 \end{aligned} \tag{4}$$

The meaning of each variable of equations 1 to 4 is described in [8, 11, 15, 12].

3 CONSTITUTIVE RELATIONSHIPS

For the gaseous mixture of dry air and water vapour, the ideal gas law is introduced. The equation of state of perfect gas (Clapeyrons equation) and Dalton's law are applied to dry air (*ga*), water vapor (*gw*) and moist air (*g*). In the partially saturated zones, the equilibrium water vapor pressure $p^{gw}(\mathbf{x}, t)$ can be obtained from the Kelvin-Laplace equation, where the water vapor saturation pressure $p^{gws}(\mathbf{x}, t)$ depending only upon the temperature, can be calculated from the Clausius-Clapeyron equation or from an empirical correlation. The saturation degree $S_\pi(\mathbf{x}, t)$ and the relative permeability $k^{r\pi}(\mathbf{x}, t)$ are experimentally determined functions. The solid skeleton is assumed elastic, homogeneous and isotropic in the numerical simulations described in Section 5.

4 SPATIAL AND TIME DISCRETIZATION

The Petrov-Galerkin method with different interpolation functions for the primary variables and the test functions is applied for the spatial discretization. Isoparametric mixed element (see. Figure 1) is used and the shape functions for each phase are

$$\begin{cases} p^g = \mathbf{N}_g \bar{\mathbf{p}}^g \\ p^c = \mathbf{N}_c \bar{\mathbf{p}}^c \\ T = \mathbf{N}_T \bar{\mathbf{T}} \\ \mathbf{u} = \mathbf{N}_u \bar{\mathbf{u}} \end{cases} \tag{5}$$

After spatial discretization, the following non-symmetric, non-linear and coupled system of equations is obtained

$$\begin{cases} \mathbf{M}_{uu} \ddot{\mathbf{u}} + \int \mathbf{B}^T \boldsymbol{\sigma}' dV - \mathbf{K}_{ug} \bar{\mathbf{p}}^g + \mathbf{K}_{uc} \bar{\mathbf{p}}^c = \mathbf{f}_u \\ \mathbf{C}_{gg} \dot{\bar{\mathbf{p}}}^g + \mathbf{C}_{gc} \dot{\bar{\mathbf{p}}}^c - \mathbf{C}_{gT} \dot{\bar{\mathbf{T}}} + \mathbf{C}_{gu} \dot{\bar{\mathbf{u}}} + \mathbf{K}_{gg} \bar{\mathbf{p}}^g - \mathbf{K}_{gc} \bar{\mathbf{p}}^c - \mathbf{K}_{gT} \bar{\mathbf{T}} = \mathbf{f}_g \\ \mathbf{C}_{cg} \dot{\bar{\mathbf{p}}}^g + \mathbf{C}_{cc} \dot{\bar{\mathbf{p}}}^c + \mathbf{C}_{cT} \dot{\bar{\mathbf{T}}} + \mathbf{C}_{cu} \dot{\bar{\mathbf{u}}} - \mathbf{K}_{cg} \bar{\mathbf{p}}^g + \mathbf{K}_{cc} \bar{\mathbf{p}}^c + \mathbf{K}_{cT} \bar{\mathbf{T}} = \mathbf{f}_c \\ -\mathbf{C}_{Tg} \dot{\bar{\mathbf{p}}}^g - \mathbf{C}_{Tc} \dot{\bar{\mathbf{p}}}^c + \mathbf{C}_{TT} \dot{\bar{\mathbf{T}}} - \mathbf{C}_{Tu} \dot{\bar{\mathbf{u}}} - \mathbf{K}_{Tg} \bar{\mathbf{p}}^g + \mathbf{K}_{Tc} \bar{\mathbf{p}}^c + \mathbf{K}_{TT} \bar{\mathbf{T}} = \mathbf{f}_T \end{cases} \tag{6}$$

The generalized Newmark method is applied to discretize in time the non-linear system equations 6 and the dynamic response at step $n+1$ for the general non-linear system is obtained, in which the unknowns are $\mathbf{X}_{n+1} = [\Delta \ddot{\mathbf{u}}, \Delta \dot{\bar{\mathbf{p}}}^g, \Delta \dot{\bar{\mathbf{p}}}^c, \Delta \dot{\bar{\mathbf{T}}}]$. The non-linear

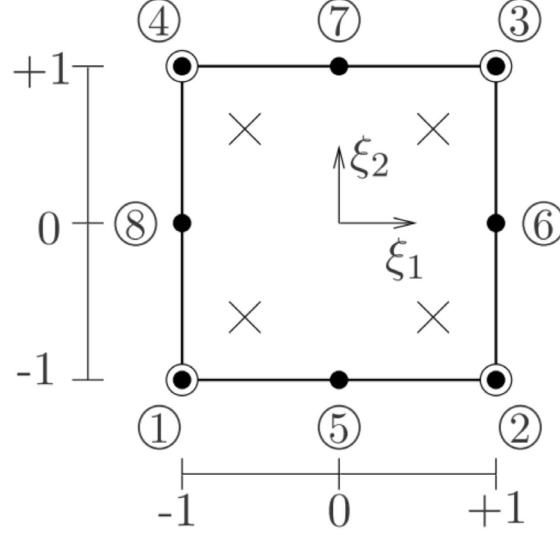


Figure 1: Isoparametric mixed element. According to the interpolation model (biquadratic for \mathbf{u} , bilinear for p^g , p^c and T), the process variables are related to different nodes: (•) solid skeleton displacements, (o) fluids pressure/temperature

system is solved by the Newton-Raphson method, written in concise form as

$$\mathbf{J}(\mathbf{X}^i)|_{n+1} [\mathbf{X}^{i+1} - \mathbf{X}^i]_{n+1} = -\mathbf{G}(\mathbf{X}^i)|_{n+1} \quad (7)$$

5 FINITE ELEMENT VALIDATION

5.1 Force vibration tests under harmonic load

In this example the response of an homogeneous and isotropic, water-saturated, poroelastic column defined in Figure 2 is analyzed under plane-strain confined compression conditions, following [21] and [22]. The domain is surrounded by impermeable, frictionless but rigid boundaries except for the loaded top side which is perfectly drained. Furthermore, 2x2 Gauss integration points were used for numerical computation. The value of the tolerance for the global iterative Newton-Rapson procedure is fixed to 10^{-4} and the maximum number of iteration inside a time step is fixed to 30. The geomechanical characteristics of the material are given in [21]. In particular, two scenarios are tested: an high permeability case, $k=10^{-2}$ m/s, and a lower permeability case, $k=10^{-5}$ m/s. The objective of this benchmark problem is to compare the dynamic finite element solution obtained with standard and mixed elements with an existing analytical solution obtained via Laplace transform [22] as reported in [21].

The vertical displacement history at the top of the column for the two cases is plotted in Figure 3, where a period equal to 0.1 second, which is the period of the harmonic load, is shown. Figure 3a displays the comparison with the analytical solution presented in [21]

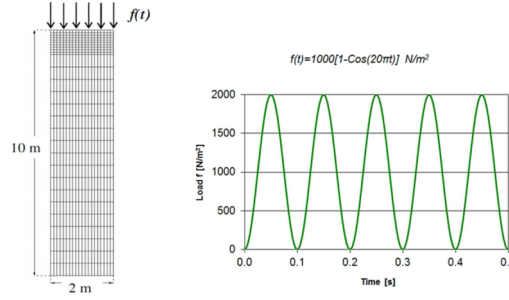


Figure 2: Description of the test model

in case of hydraulic permeability of $k=10^{-2}$ m/s and a good convergence is observed. With the lower permeability, the development of the vertical displacement of the top surface is lower than the case with higher permeability. Moreover, for permeability $k=10^{-5}$ m/s, Figure 3b shows a diversity in the accuracy of the solution when the mixed element formulation is used.

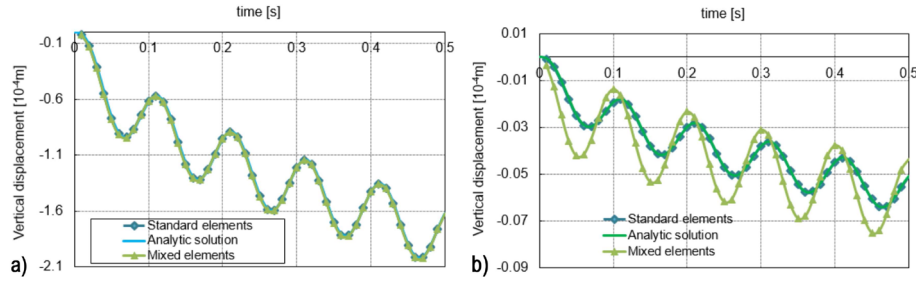


Figure 3: Top displacement history for a) $k=10^{-2}$ m/s, b) $k=10^{-5}$ m/s

5.2 Non-isothermal water saturated consolidation problem

The test problem concerning the fully saturated thermo-elastic consolidation defined in Figure 4 is now solved and its solution is compared with the solution of quasi-static and dynamic problems (with standard elements) done in previous work [23, 18]. The plane-strain medium is confined in a column of 7 m height and 2 m width and subjected to a surface loading 10 kPa and a surface temperature jump of 50 K above the initial temperature of 293.15 K.

The geomechanical characteristics of the Aboustis material are given in [23, 13] with permeability: $k = 2.55 \cdot 10^{-7}$ m/s. The liquid water and the solid grain are assumed incompressible for the static analysis, whereas the compressibility of the liquid water is taken into account in the dynamic analysis.

The initial variables for whole node are the liquid water pressure in hydrostatic condition, the air pressure at atmospheric pressure and the temperature at 293.15 K. The

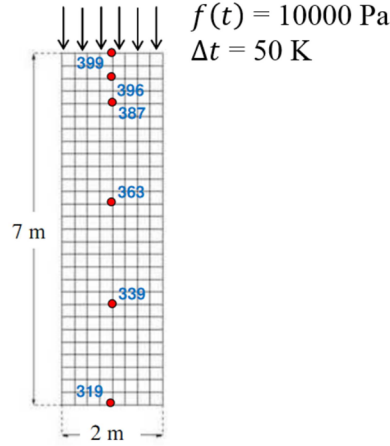


Figure 4: Description of the Aboustit test model

boundary conditions of this problems are as follows: the bottom and lateral surface are impervious to both water and air; the top boundary is drained. The normal and tangential displacements are constrained at bottom; at lateral surface, the normal displacement is constrained. 2x2 Gauss point integration scheme was used. During the computation, the value of the tolerance for the global iterative Newton-Rapson procedure is fixed to 10^{-4} and the maximum number of iteration inside a time step is fixed to 30.

Figure 5 presents the comparison between the quasi-static solution with standard elements, the dynamic solution with standard elements and the dynamic solution with mixed elements for the nodes 319 and 399 displayed in Figure 4. It is observed a difference in the solution with mixed elements for vertical displacements and capillary pressure, while the solution of temperature is not affected by the element type used for the computations.

6 CONCLUSIONS

A mixed finite element formulation for the dynamic analysis of hydro-thermo-mechanical behavior of water saturated and partially saturated porous media has been presented. The model has been implemented in the finite element code Comes-Geo [8, 11, 12, 13, 14, 15, 16, 17, 18] developed at the University of Padova, Department of Civil, Environmental and Architectural Engineering. The comparison between the finite element solution obtained with standard elements and mixed elements in dynamics has been presented, showing that the use of quadratic-linear Taylor-Hood elements for the monolithic treatment of dynamics results less accurate with respect the case of applied equal-order approximation when low permeability is considered. This aspect deserves further investigations.

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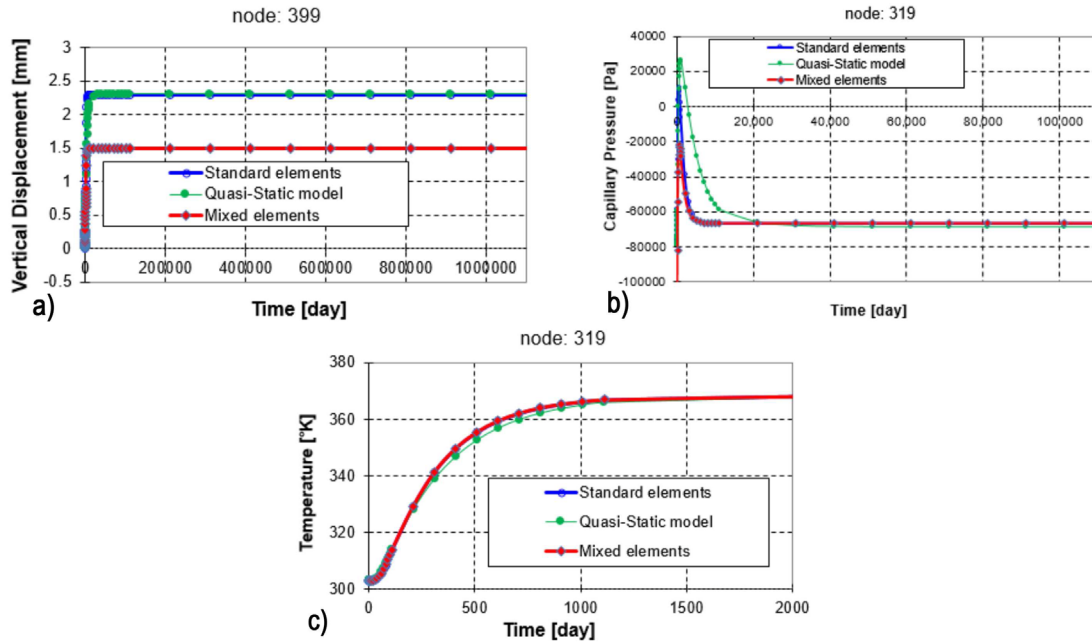


Figure 5: Time-history variables at node references; a) vertical displacement at top sample, b) capillary pressure at bottom sample and c) temperature at bottom sample

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